Free Energy and Phase Equilibria

Thermodynamic Integration (7.1)
Chemical Potentials (7.2)
Overlapping Distributions (7.2)
Umbrella Sampling (7.4)
Application: Phase Diagram of Carbon

Why Free Energies?

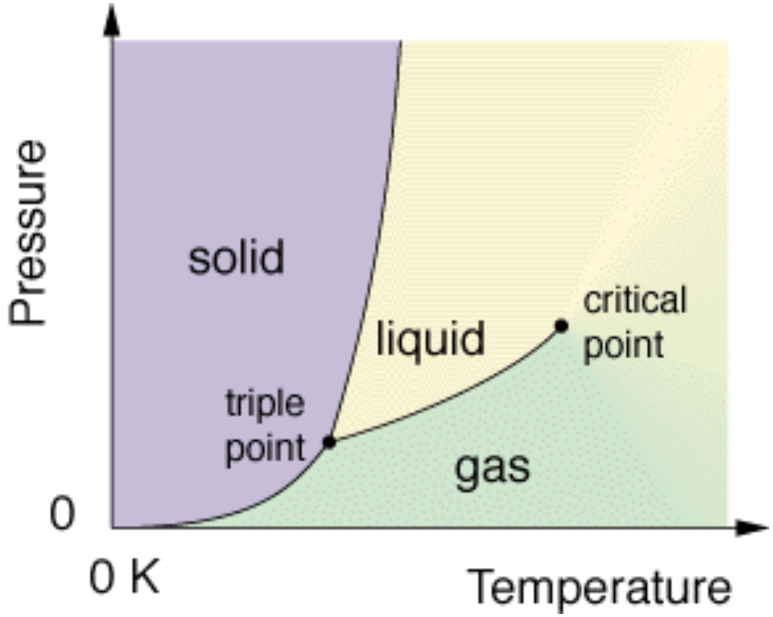
- Reaction equilibrium constants $A \leftrightarrow B$

$$K = \frac{[B]}{[A]} = \frac{p_B}{p_A} = \exp\left[-\beta(G_B - G_A)\right]$$

- Examples:
  - Chemical reactions: e.g. catalysis, etc....
  - Protein folding, binding: free energy gives binding constants

- Phase diagrams
  - Prediction of thermodynamic stability of phases
  - Coexistence lines
  - Critical points
  - Triple points
  - First order/second order phase transitions

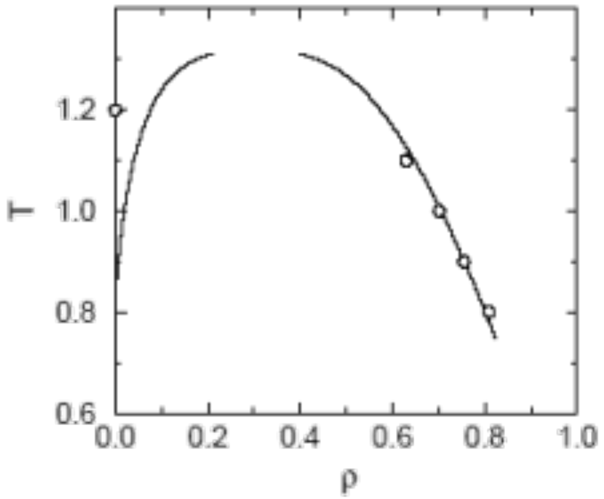
Phase diagrams



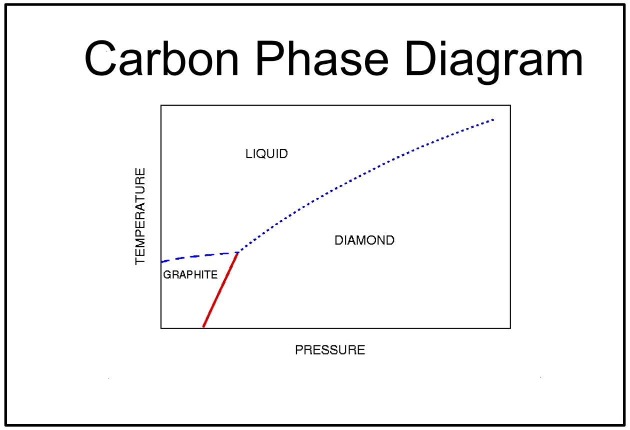
Critical point: no transition between liquid and vapor

Triple point: liquid, vapor and solid in equilibrium.

How do we compute these lines?



Along the liquid-gas coexistence line increasing the pressure and temperature at constant volume the liquid density becomes lower and the vapor density higher.


Carbon Phase Diagram

Phase equilibrium

Criteria for equilibrium (for single component)

$$T_I = T_{II} \qquad P_I = P_{II} \qquad \mu_I = \mu_{II}$$

Chemical potential

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{V,T} = \left(\frac{\partial G}{\partial N}\right)_{P,T} = G_m$$

If $\mu_I > \mu_{II}$: transport of particles from phase I to phase II.

Stable phase:

Lowest chemical potential (for single phase: lowest Gibbs free energy)

Relation thermodynamic potentials

Helmholtz free energy: $F = U - TS$

Gibbs free energy: $G = F + PV$

Suppose we have F(n,V,T)

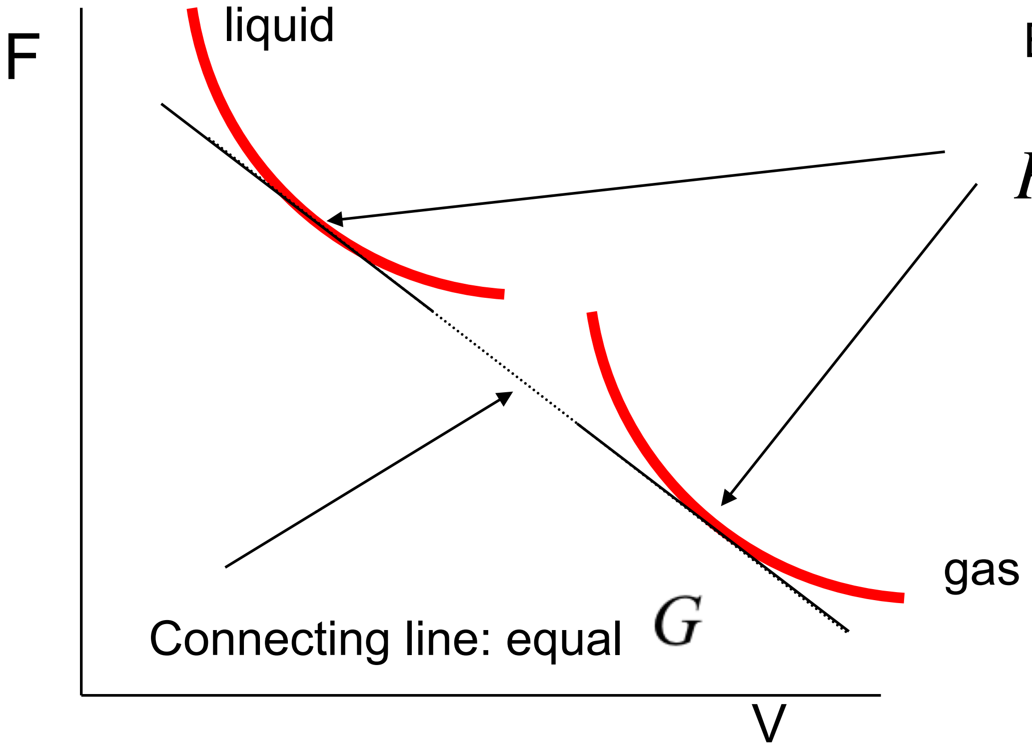
Then we can find G from F from:

$$P = -\left(\frac{\partial F}{\partial V}\right)_{n,T} \qquad G = F - \left(\frac{\partial F}{\partial V}\right)_{n,T} V$$

All thermodynamic quantities can be derived from F and its derivatives

Phase equilibria from F(V,T)

Common tangent construction



Equal tangents

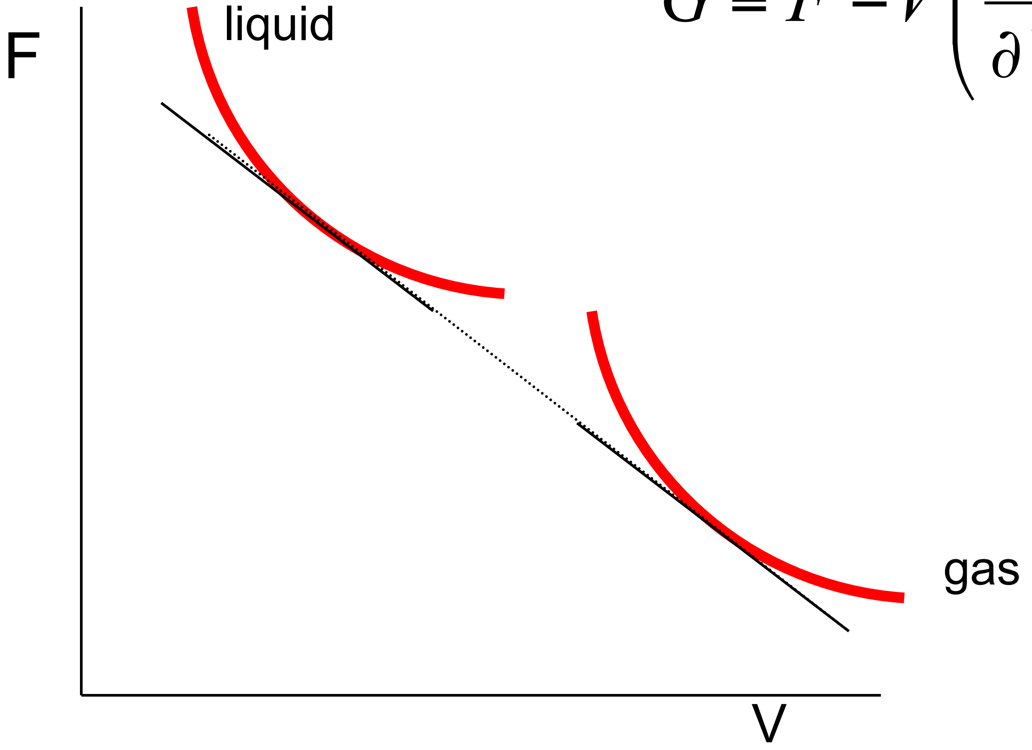
$$P = -\left(\frac{\partial F}{\partial V}\right)_{n,T}$$

Connecting line: equal G

Common tangent construction

Helmholtz Free Energy Perspective

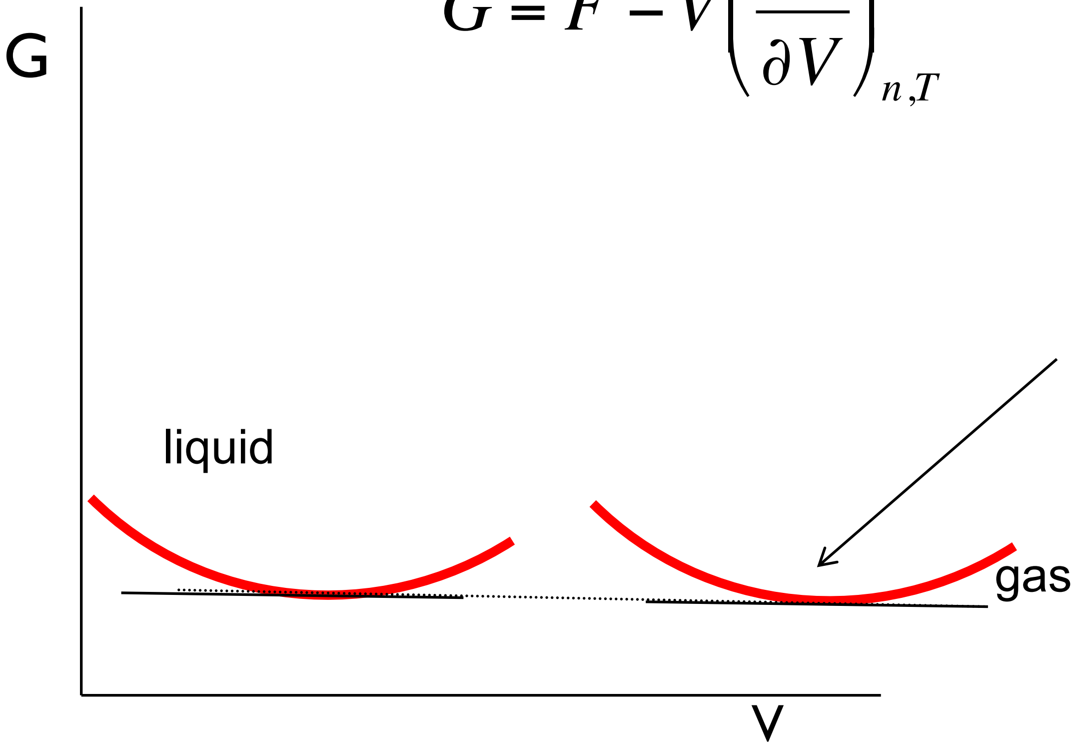
$$G = F - V\left(\frac{\partial F}{\partial V}\right)_{n,T}$$



Common tangent construction

Gibbs Free Energy Perspective

$$G = F - V\left(\frac{\partial F}{\partial V}\right)_{n,T}$$



Both liquid and vapor G equal and minimal

Only equilibrium when P,T is on coexistence line.

We need F or μ

- So equilibrium from F(V) alone or from P and G (or μ)

$$F(V) = F(V_0) + \int_{V_0}^{V} \left(\frac{\partial \mathrm{F}}{\partial \mathrm{V}}\right)_{\mathrm{N,T}} \mathrm{dV} = F(V_0) - \int P\mathrm{dV}$$

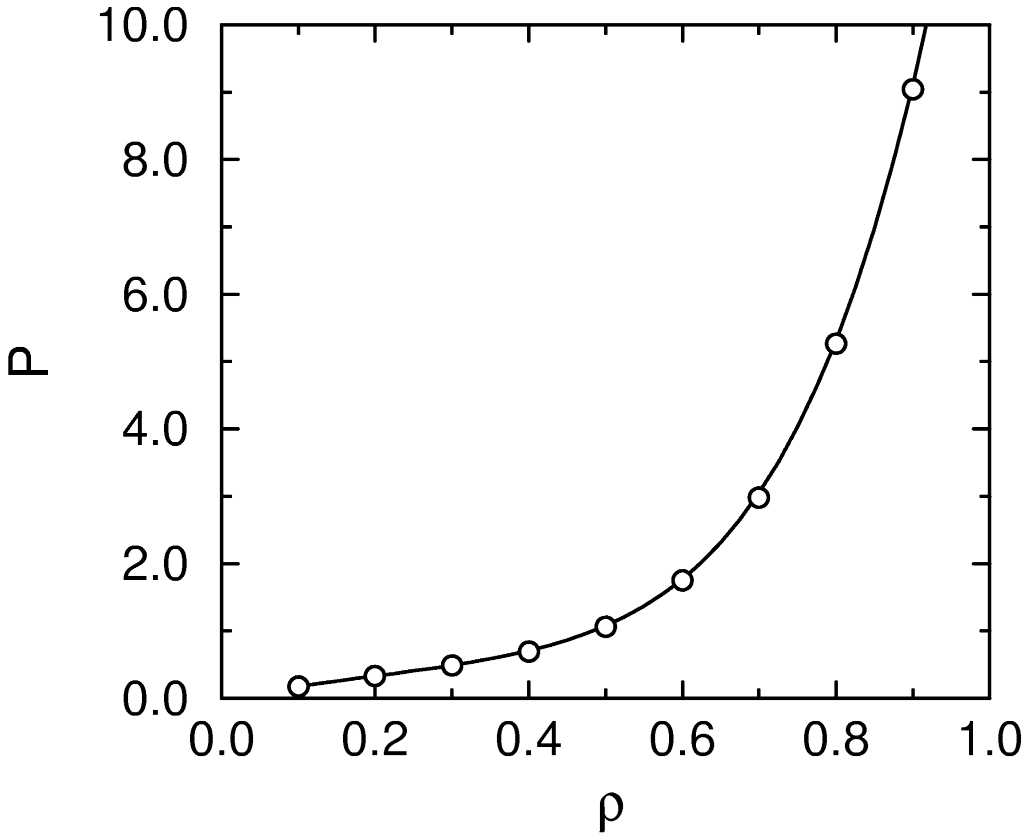
$$F(\rho) = F(\rho_0) + N\int_{\rho_0}^{\rho} \frac{P(\rho')}{\rho^2}\mathrm{d}\rho'$$

- So in fact for only 1 point of the equation of state the F is needed

- For liquid e.o.s even from ideal gas

$$\beta F(\rho)/N = \beta F^{id}(\rho)/N + \int_0^{\rho} \frac{\beta P(\rho') - \rho'}{\rho^2}\mathrm{d}\rho'$$

Equation of state



$$P = P(\rho, T)$$

$$\left(\frac{\partial F}{\partial V}\right)_{N,T} = -P$$

$$F(\rho) = F(\rho_0) + N\int_{\rho_0}^{\rho} \frac{P(\rho')}{\rho^2} \mathrm{d}\rho'$$

$$\boxed{\beta F(\rho)/N = \beta F^{id}(\rho)/N + \int_0^{\rho} \frac{\beta P(\rho') - \rho'}{\rho^2} \mathrm{d}\rho'}$$

Free Energies and Phase Equilibria

General Strategies

- Determine free energy of both phases relative to a reference state

  Free energy difference calculation

  General applicable: Gas, Liquid, Solid, Inhomogeneous systems, …

- Determine free energy difference between two phases

  Gibbs Ensemble (Lecture Thijs Vlugt)

  Specific applicable: Gas, Liquid

Statistical Thermodynamics

Probability to find a particular configuration (NVT)

$$P\left(\mathrm{r}^N\right) = \frac{1}{Q_{NVT}} \frac{1}{\Lambda^{3N} N!} \int \mathrm{dr}'^N \delta\left(\mathrm{r}'^N - \mathrm{r}^N\right) \exp\left[-\beta U\left(\mathrm{r}'^N\right)\right] \propto \exp\left[-\beta U\left(\mathrm{r}^N\right)\right]$$

Partition function

$$Q_{NVT} = \frac{1}{\Lambda^{3N} N!} \int \mathrm{dr}^N \exp\left[-\beta U\left(\mathrm{r}^N\right)\right]$$

Free energy

$$\beta F = -\ln\left(Q_{NVT}\right)$$

Ensemble average

$$\langle A \rangle_{NVT} = \frac{1}{Q_{NVT}} \frac{1}{\Lambda^{3N} N!} \int \mathrm{dr}^N A\left(\mathrm{r}^N\right) \exp\left[-\beta U\left(\mathrm{r}^N\right)\right]$$

Ensemble average versus free energy

Generate configuration using MC: $\left\{\mathrm{r}_1^N, \mathrm{r}_2^N, \mathrm{r}_3^N, \mathrm{r}_4^N \cdots, \mathrm{r}_M^N\right\}$

$$\overline{A} = \frac{1}{M}\sum_{i=1}^{M} A\left(\mathrm{r}_i^N\right) \approx \frac{\int \mathrm{dr}^N A\left(\mathrm{r}^N\right)\exp\left[-\beta U\left(\mathrm{r}^N\right)\right]}{\int \mathrm{dr}^N \exp\left[-\beta U\left(\mathrm{r}^N\right)\right]} = \left\langle A\right\rangle_{NVT}$$

Generate configuration using MD: $\left\{\mathrm{r}_1^N, \mathrm{r}_2^N, \mathrm{r}_3^N, \mathrm{r}_4^N \cdots, \mathrm{r}_M^N\right\}$

$$\overline{A} = \frac{1}{M}\sum_{i=1}^{M} A\left(\mathrm{r}_i^N\right) \approx \frac{1}{T}\int_0^T dt A(t) \int \approx \left\langle A\right\rangle_{NVT}$$

ergodicity

$$\beta F = -\ln Q_{NVT} = -\ln \frac{1}{\Lambda^{3N} N!}\int \mathrm{dr}^N \exp\left[-\beta U\left(\mathrm{r}^N\right)\right]$$

F is difficult, because requires accounting of phase space volume

I - Thermodynamic integration

- Known reference state λ=0
- Unknown target state λ=1

Reference System

Coupling parameter

Target System

$$U(\lambda) = (1-\lambda)U_I + \lambda U_{II}$$

$$Q_{NVT}(\lambda) = \frac{1}{\Lambda^{3N} N!}\int \mathrm{d}\mathrm{r}^N \exp\left[-\beta U(\lambda)\right]$$

$$F(\lambda=1) - F(\lambda=0) = \int_{\lambda=0}^{\lambda=1} d\lambda \left(\frac{\partial F(\lambda)}{\partial \lambda}\right)_{N,V,T}$$

Thermodynamic integration

$$Q_{NVT}(\lambda) = \frac{1}{\Lambda^{3N} N!} \int \mathrm{d}\mathbf{r}^N \exp\left[-\beta U(\lambda)\right]$$

$$\left(\frac{\partial F(\lambda)}{\partial \lambda}\right)_{N,T} = -\frac{1}{\beta}\frac{\partial}{\partial \lambda}\ln(Q) = -\frac{1}{\beta}\frac{1}{Q}\frac{\partial Q}{\partial \lambda}$$

$$= \frac{\int \mathrm{d}\mathbf{r}^N \left(\partial U(\lambda)/\partial \lambda\right)\exp\left[-\beta U(\lambda)\right]}{\int \mathrm{d}\mathbf{r}^N \exp\left[-\beta U(\lambda)\right]}$$

$$= \left\langle \frac{\partial U(\lambda)}{\partial \lambda} \right\rangle_{\lambda}$$

Free energy as ensemble average!

$$F(\lambda = 1) - F(\lambda = 0) = \int_{\lambda=0}^{\lambda=1} d\lambda \left(\frac{\partial F(\lambda)}{\partial \lambda}\right)_{N,V,T}$$

$$F(\lambda = 1) - F(\lambda = 0) = \int \mathrm{d}\lambda \left\langle \frac{\partial U(\lambda)}{\partial \lambda} \right\rangle_{\lambda}$$

Example

- In general

$$U(\lambda) = (1-\lambda)U_I + \lambda U_{II}$$

$$\left\langle \frac{\partial U(\lambda)}{\partial \lambda} \right\rangle_\lambda = \langle U_{II} - U_I \rangle_\lambda$$

- Specific example

$$U(\lambda) = U^{LJ} + \lambda U^{\text{dipole-dipole}}$$

$$U(0) = U^{LJ}$$ Lennard-Jones

$$U(1) = U^{\text{Stockm}}$$ Stockmayer

$$\left\langle \frac{\partial U(\lambda)}{\partial \lambda} \right\rangle_\lambda = \langle U^{dip-dip} \rangle_\lambda$$

Free energy of solid

More difficult. What is reference?

Not the ideal gas.

Instead it is the Einstein crystal: harmonic oscillators around r_0

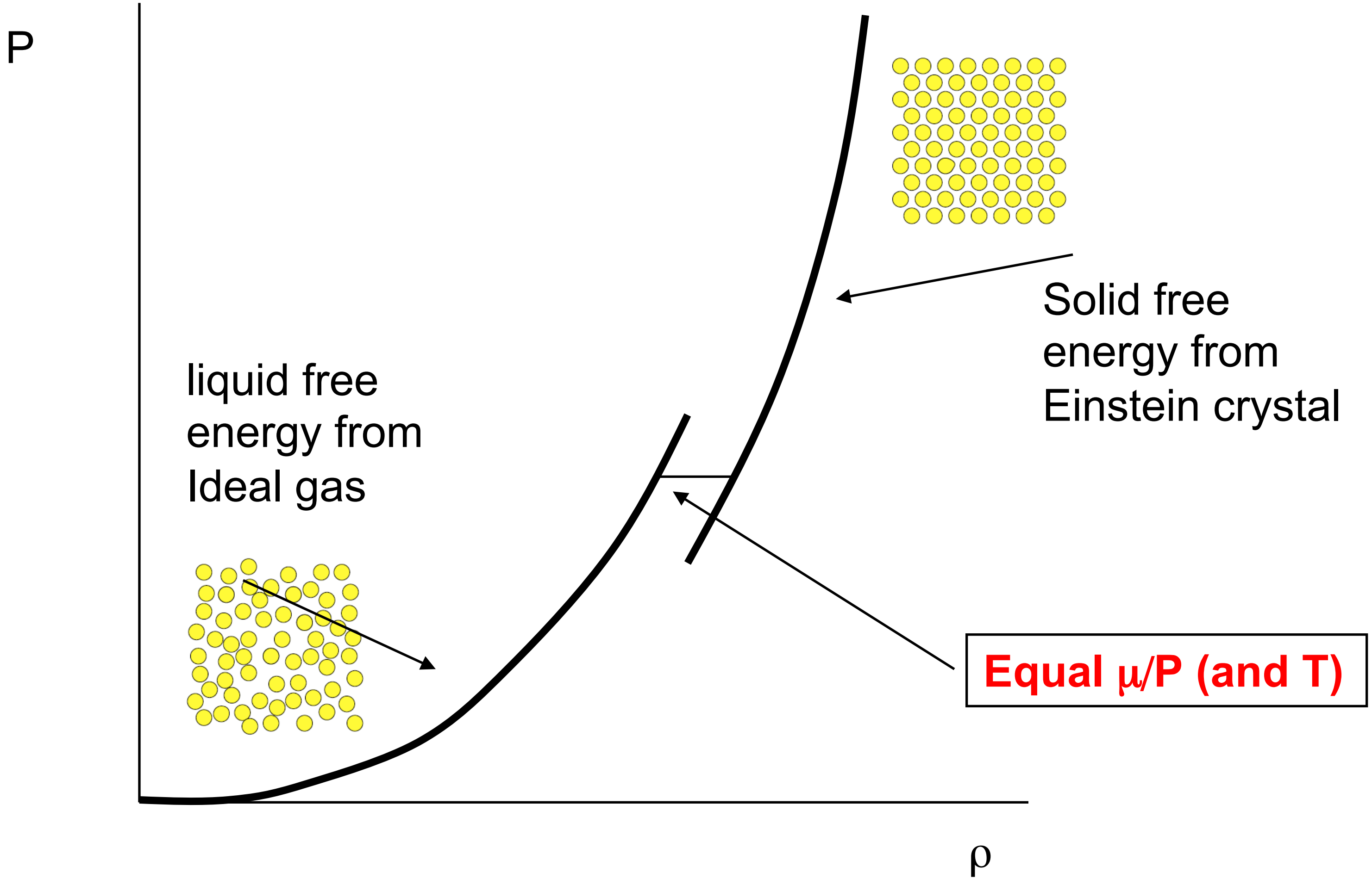
$$U\left(\lambda;r^N\right)=(1-\lambda)U\left(r^N\right)+\lambda U\left(r_0^N\right)+\lambda\sum_{i=1}^{N}\alpha(r_i-r_i)^2$$

$$F=F_{ein}-\int_{\lambda=0}^{\lambda=1}d\lambda\left\langle\frac{\partial U(\lambda)}{\partial\lambda}\right\rangle_\lambda$$

Note, here:
λ = 1 Reference System
λ = 0 Target System

$$F=F_{ein}-\int_{\lambda=0}^{\lambda=1}d\lambda\left\langle -U\left(r^N\right)+U\left(r_0^N\right)+\sum_{i=1}^{N}\alpha(r_i-r_i)^2\right\rangle_\lambda$$

Hard sphere freezing



liquid free
energy from
Ideal gas

Solid free
energy from
Einstein crystal

Equal μ/P (and T)

II - Thermodynamic perturbation

Two systems:

System 0: N, V, T, U_0

System 1: N, V, T, U_1

$$Q_0 = \frac{V^N}{\Lambda^{3N} N!} \int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)$$

$$Q_1 = \frac{V^N}{\Lambda^{3N} N!} \int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_1\right)$$

$$\begin{aligned}
\Delta\beta F = \beta F_1 - \beta F_0 &= -\ln\left(Q_1/Q_0\right) \\
&= -\ln \frac{\int \mathrm{d}\mathbf{s}^N \exp\left[-\beta U_1\right]}{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)} \\
&= -\ln \frac{\int \mathrm{d}\mathbf{s}^N \exp\left[-\beta\left(U_1 - U_0\right)\right]\exp\left[-\beta U_0\right]}{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)}
\end{aligned}$$

$$\Delta\beta F = -\ln\left\langle \exp\left[-\beta\left(U_1 - U_0\right)\right]\right\rangle_0$$

Chemical potential

$$Q_{NVT} = \frac{V^N}{\Lambda^{3N} N!} \int \mathrm{d}s^N \exp\left[-\beta U\left(s^N; L\right)\right]$$

$$\begin{aligned}\beta F &= -\ln\left(Q_{NVT}\right) \\ &= -\ln\left(\frac{V^N}{\Lambda^{3N} N!}\right) - \ln\left(\int \mathrm{d}s^N \exp\left[-\beta U\left(s^N; L\right)\right]\right) \\ &= -N \ln\left(\frac{1}{\Lambda^3 \rho}\right) + N - \ln\left(\int \mathrm{d}s^N \exp\left[-\beta U\left(s^N; L\right)\right]\right)\end{aligned}$$

$$\left.\begin{array}{l}\beta F = \beta F^{IG} + \beta F^{ex} \\[1em] \boxed{\mu \equiv \left(\frac{\partial F}{\partial N}\right)_{V,T}}\end{array}\right\} \begin{array}{l}\beta\mu = \beta\mu^{IG} + \beta\mu^{ex} \\[1em] \beta\mu^{IG} \equiv \left(\frac{\partial \beta F^{IG}}{\partial N}\right)_{V,T} \quad \beta\mu^{ex} \equiv \left(\frac{\partial \beta F^{ex}}{\partial N}\right)_{V,T}\end{array}$$

Widom test particle insertion

$$\beta\mu \equiv \left(\frac{\partial \beta F}{\partial N}\right)_{V,T}$$

$$\beta\mu = \frac{\beta F(N+1)) - \beta F(N))}{N+1-N}$$

$$= -\ln\frac{Q(N+1)}{Q(N)}$$

$$= -\ln\left(\frac{\dfrac{V^{N+1}}{\Lambda^{3N+3}(N+1)!}}{\dfrac{V^{N}}{\Lambda^{3N}N!}}\right) - \ln\left(\frac{\int \mathrm{ds}^{N+1}\exp\left[-\beta U\left(\mathrm{s}^{N+1};L\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

$$= -\ln\left(\frac{V}{\Lambda^{3}(N+1)}\right) - \ln\left(\frac{\int \mathrm{ds}^{N+1}\exp\left[-\beta U\left(\mathrm{s}^{N+1};L\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

$$\beta\mu = \beta\mu^{IG} + \beta\mu^{ex}$$

$$\beta\mu^{ex} = -\ln\left(\frac{\int \mathrm{ds}^{N+1}\exp\left[-\beta U\left(\mathrm{s}^{N+1};L\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

Widom test particle insertion

$$\beta\mu^{ex} = -\ln\left(\frac{\int \mathrm{ds}^{N+1}\exp\left[-\beta U\left(\mathrm{s}^{N+1};L\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

$$U\left(\mathrm{s}^{N+1};L\right) = \Delta U^{+} + U\left(\mathrm{s}^{N};L\right)$$

$$\beta\mu^{ex} = -\ln\left(\frac{\int \mathrm{ds}^{N}\int \mathrm{ds}_{N+1}\exp\left[-\beta\left(\Delta U^{+} + U\left(\mathrm{s}^{N};L\right)\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

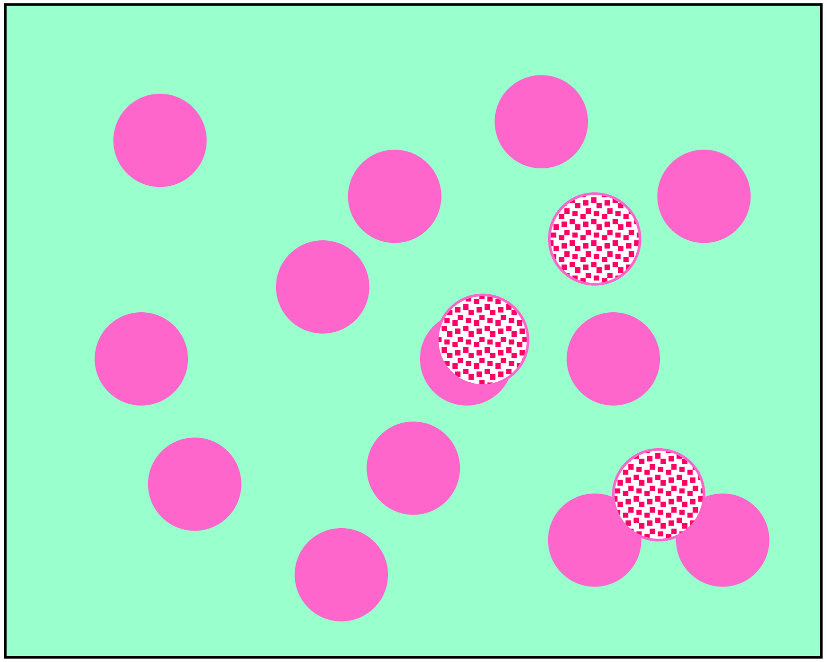
$$= -\ln\left(\frac{\int \mathrm{ds}_{N+1}\int \mathrm{ds}^{N}\left\{\exp\left[-\beta\Delta U^{+}\right]\right\}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}{\int \mathrm{ds}^{N}\exp\left[-\beta U\left(\mathrm{s}^{N};L\right)\right]}\right)$$

$$\boxed{= -\ln\left(\int \mathrm{ds}_{N+1}\left\langle\exp\left[-\beta\Delta U^{+}\right]\right\rangle_{NVT}\right)}$$

Ghost particle!

Hard spheres

$$\beta\mu^{ex} = -\ln\left(\int \mathrm{d}s_{N+1} \left\langle \exp\left[-\beta\Delta U^{+}\right]\right\rangle_{NVT}\right)$$



$$U(r) = \begin{cases} \infty & r \le \sigma \\ 0 & r > \sigma \end{cases}$$

$$\exp\left[-\beta\Delta U^{+}\right] = \begin{cases} 0 & \text{if overlap} \\ 1 & \text{no overlap} \end{cases}$$

$\left\langle \exp\left[-\beta\Delta U^{+}\right]\right\rangle$ probability to insert a test particle!

But, … may fail at high density

III - Overlapping Distribution Method

Two systems:

System 0: $N,\ V, T,\ U_0$

System 1: $N,\ V, T,\ U_1$

$$Q_0 = \frac{V^N}{\Lambda^{3N} N!} \int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right) \qquad Q_1 = \frac{V^N}{\Lambda^{3N} N!} \int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_1\right)$$

$$\Delta\beta F = \beta F_1 - \beta F_0 = -\ln\left(Q_1/Q_0\right) = -\ln\left(\frac{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_1\right)}{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)}\right) = -\ln\left(\frac{Q_1}{Q_0}\right)$$

$$p_0(\Delta U) = \frac{\int \mathrm{d}\mathbf{s}^N \exp(-\beta U_0)\delta(U_1 - U_0 - \Delta U)}{\int \mathrm{d}\mathbf{s}^N \exp(-\beta U_0)} \qquad p_1(\Delta U) = \frac{\int \mathrm{d}\mathbf{s}^N \exp(-\beta U_1)\delta(U_1 - U_0 - \Delta U)}{\int \mathrm{d}\mathbf{s}^N \exp(-\beta U_1)}$$

=ΔU (δ function)

$$p_1(\Delta U) = \frac{\int \mathrm{d}\mathbf{s}^N \exp\left[-\beta(U_1 - U_0)\right]\exp\left[-\beta U_0\right]\delta(U_1 - U_0 - \Delta U)}{\int \mathrm{d}\mathbf{s}^N \exp(-\beta U_1)}$$

$$= \frac{1}{Q_1} = \frac{Q_0}{Q_1}\frac{1}{Q_0}$$

$$\frac{Q_0}{Q_1} = \exp\left(\beta \Delta F\right)$$

$$= \frac{Q_0}{Q_1}\exp\left(-\beta \Delta U\right)\frac{\int \mathrm{d}\mathbf{s}^N \exp\left[-\beta U_0\right]\delta\left(U_1 - U_0 - \Delta U\right)}{Q_0}$$

$$p_1\left(\Delta U\right) = \frac{Q_0}{Q_1}\exp\left(-\beta \Delta U\right) p_0\left(\Delta U\right)$$

$$\boxed{\ln p_1\left(\Delta U\right) = \beta\left(\Delta F - \Delta U\right) + \ln p_0\left(\Delta U\right)}$$

Overlapping Distribution Method

$$\ln p_1(\Delta U) = \beta(\Delta F - \Delta U) + \ln p_0(\Delta U)$$

$$f_0(\Delta U) \equiv \ln p_0(\Delta U) - 0.5\beta\Delta U$$

$$f_1(\Delta U) \equiv \ln p_1(\Delta U) + 0.5\beta\Delta U$$

Simulate system 0: compute f_0
Simulate system 1: compute f_1

$$\beta\Delta F = f_1(\Delta U) - f_0(\Delta U)$$

Chemical potential

System 0: *N-1, V, T, U* + 1 ideal gas

System 1: *N, V, T, U*

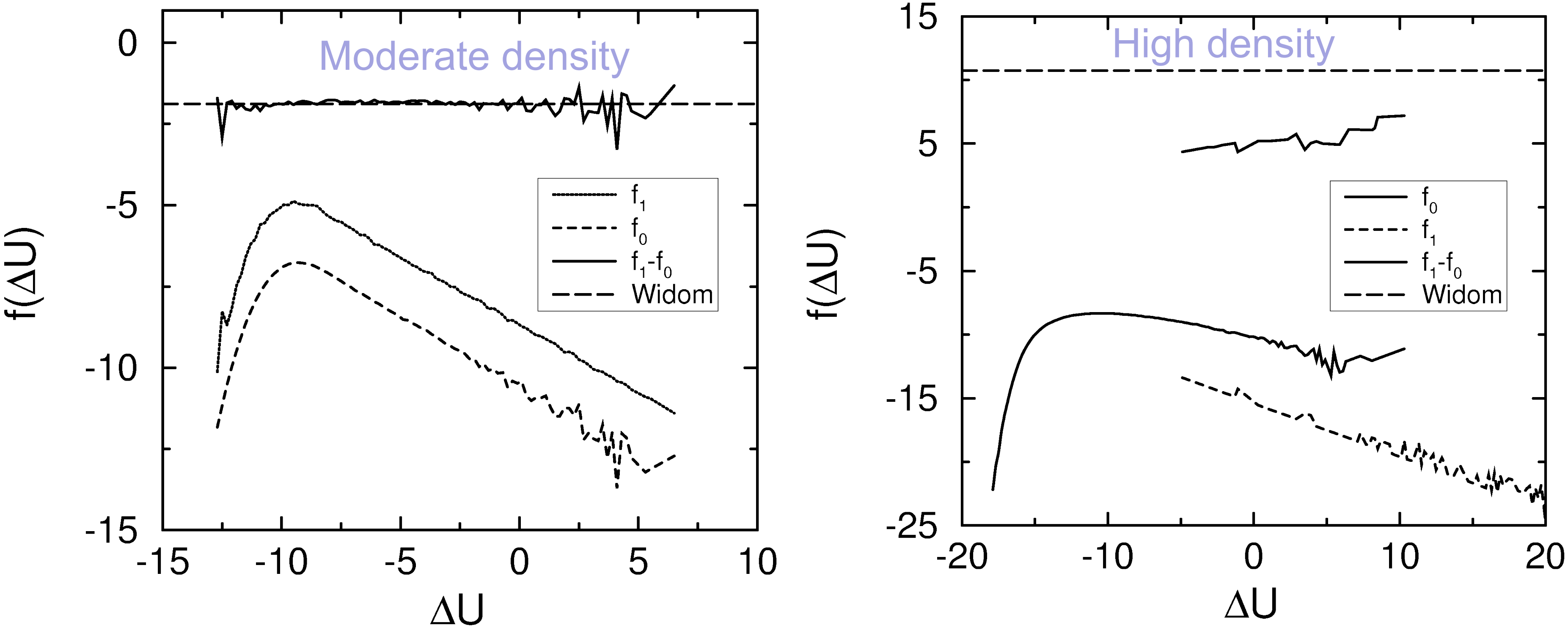
$$\Delta\beta F = \beta F_1 - \beta F_0 \equiv \beta\mu^{ex}$$

$$\Delta U = U_1 - U_0$$

System 0: test particle energy

System 1: real particle energy

$$\beta\mu^{ex} = f_1(\Delta U) - f_0(\Delta U)$$



Umbrella sampling

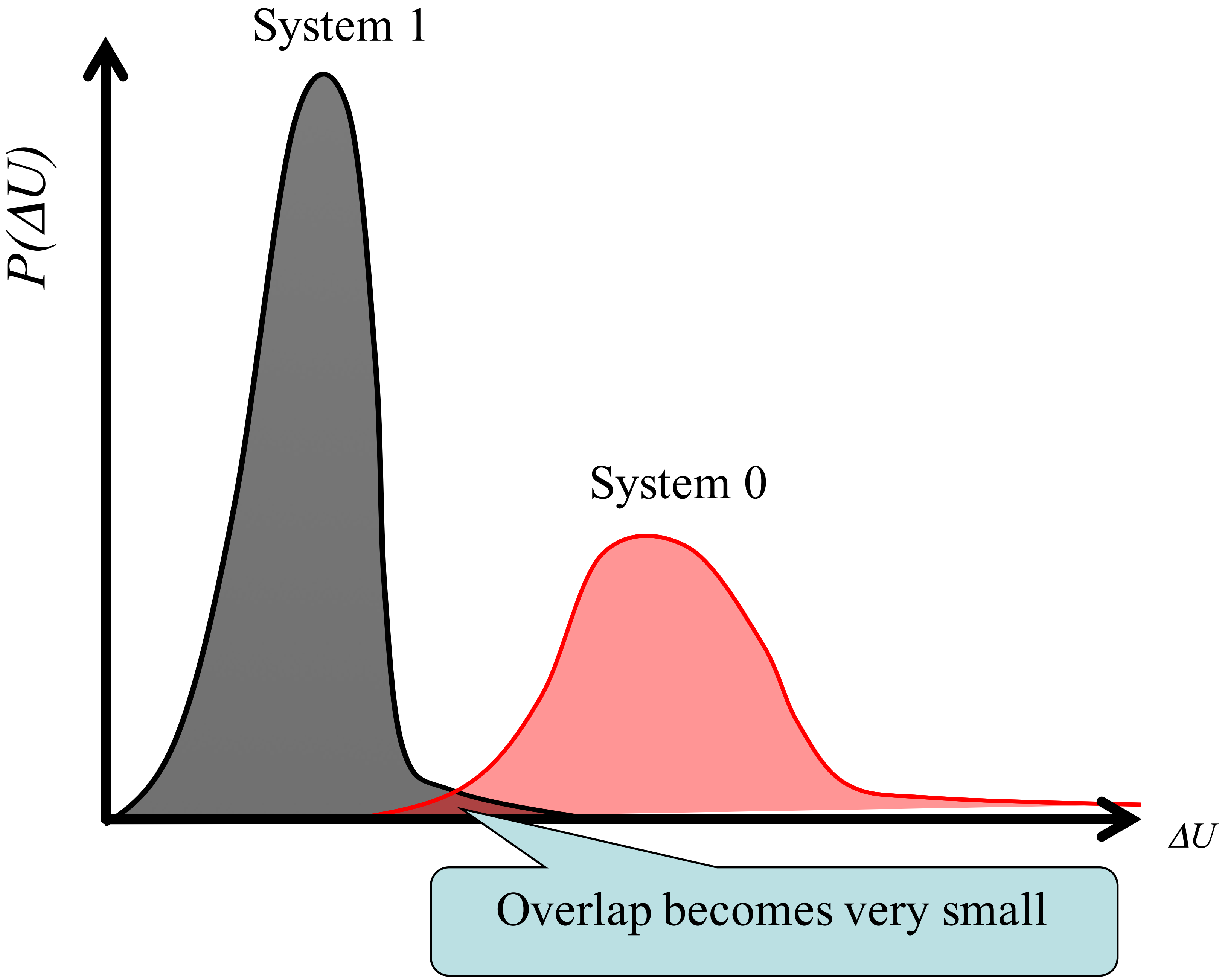
- Start with thermodynamic perturbation

$$\Delta\beta F = -\ln\left(Q_1/Q_0\right) \qquad = -\ln\left(\frac{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_1\right)}{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)}\right)$$

$$\exp\left(-\beta\Delta F\right) = \left(\frac{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)\exp\left(-\beta\Delta U\right)}{\int \mathrm{d}\mathbf{s}^N \exp\left(-\beta U_0\right)}\right)$$

$$\exp\left(-\beta\Delta F\right) = \left\langle \exp\left(-\beta\Delta U\right)\right\rangle_0$$

Can we use this for free energy difference between arbitrary systems?



Overlap becomes very small

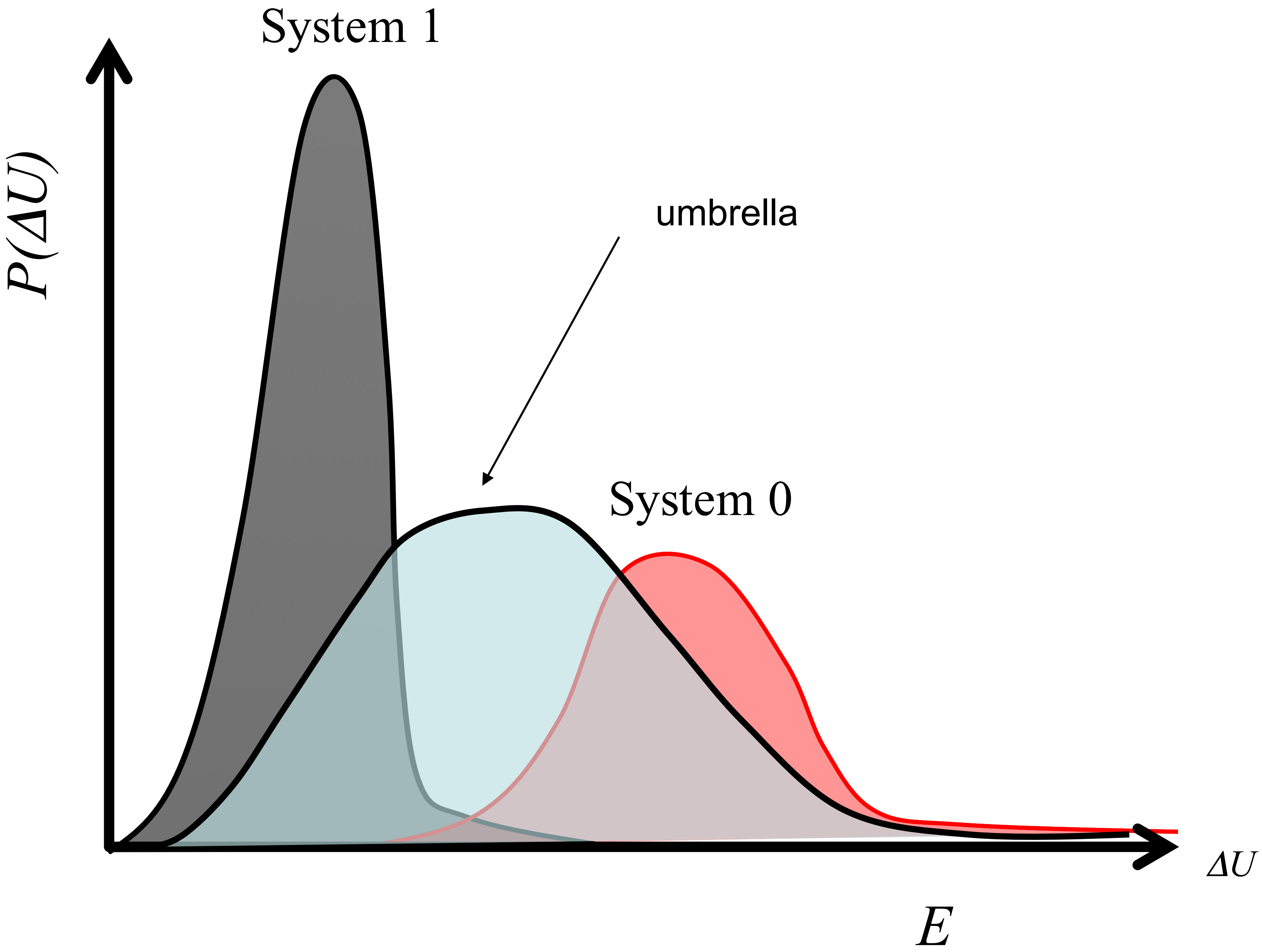
Bridging function

- Introduce function π(s^N) altering distribution.

$$\exp(-\beta\Delta F) = \left(\frac{\int d\mathbf{s}^N \pi(\mathbf{s}^N) \exp(-\beta U_1)/\pi(\mathbf{s}^N)}{\int d\mathbf{s}^N \pi(\mathbf{s}^N) \exp(-\beta U_0)/\pi(\mathbf{s}^N)} \right)$$

$$\exp(-\beta\Delta F) = \frac{\left\langle \exp(-\beta U_1)/\pi \right\rangle_\pi}{\left\langle \exp(-\beta U_0)/\pi \right\rangle_\pi}$$

- This approach is called umbrella sampling

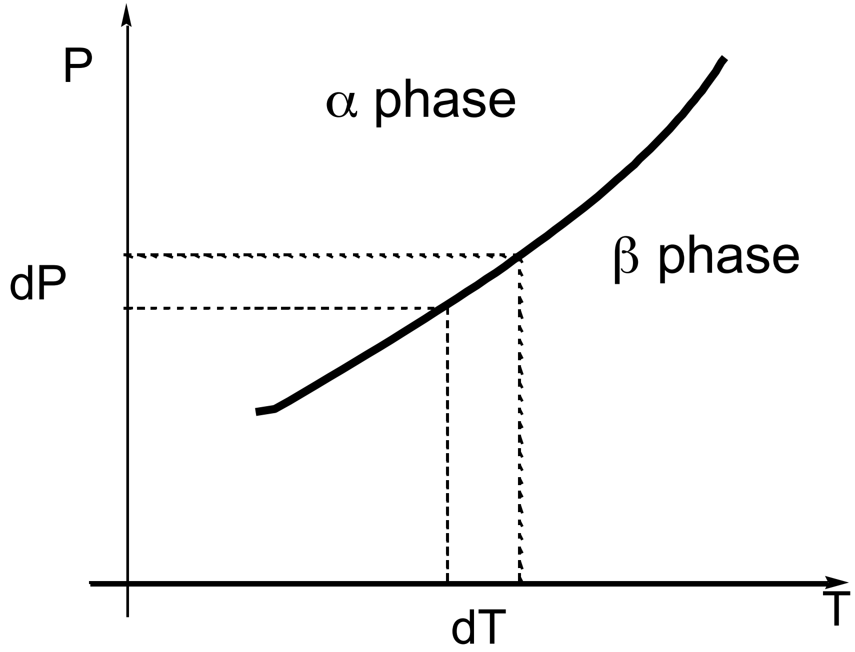

System 1
umbrella
System 0
$P(\Delta U)$
ΔU
E

Tracing coexistence curves

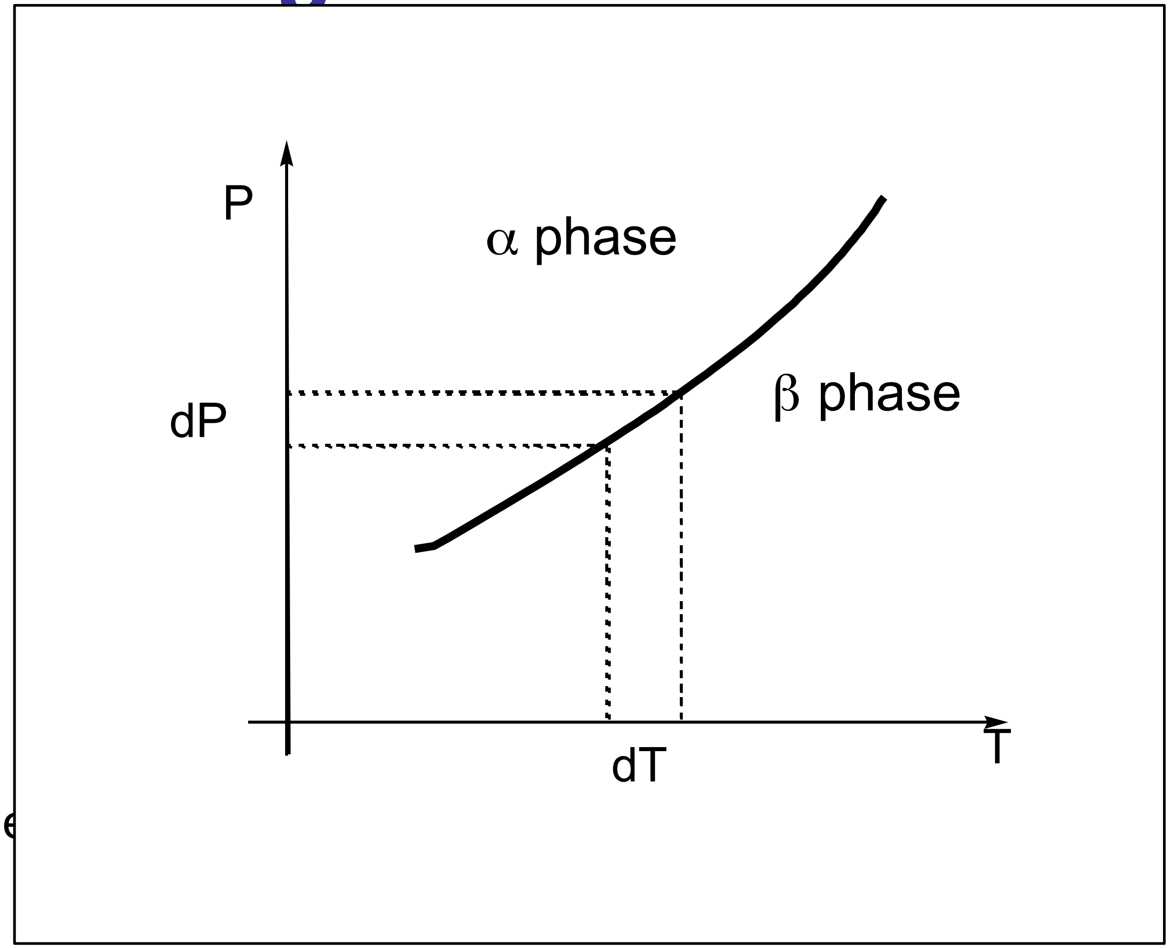
- If we have a coexistence point on the phase diagram we can integrate allong the line while maintaining coexistence.

P en T are equal along coexistence line

$d\mu_\alpha \quad = \quad d\mu_\beta$



Tracing coexistence curves



Clapyeron e

$$\frac{dP}{dT} = \frac{\Delta(U + PV)}{T\Delta V}$$

$$\boxed{dP = \frac{\Delta(U + PV)}{T\Delta V}\,dT}$$

IV - Non-Boltzmann sampling

T_1 is arbitrary!

$$\boxed{\langle A\rangle_{NVT_1}} = \frac{1}{Q_{NVT_1}}\frac{1}{\Lambda^{3N}N!}\int \mathrm{d}\mathrm{r}^N A\left(\mathrm{r}^N\right)\exp\left[-\beta_1 U\left(\mathrm{r}^N\right)\right]$$

$$= \frac{\int \mathrm{d}\mathrm{r}^N A\left(\mathrm{r}^N\right)\exp\left[-\beta_1 U\left(\mathrm{r}^N\right)\right]}{\int \mathrm{d}\mathrm{r}^N \exp\left[-\beta_1 U\left(\mathrm{r}^N\right)\right]}$$

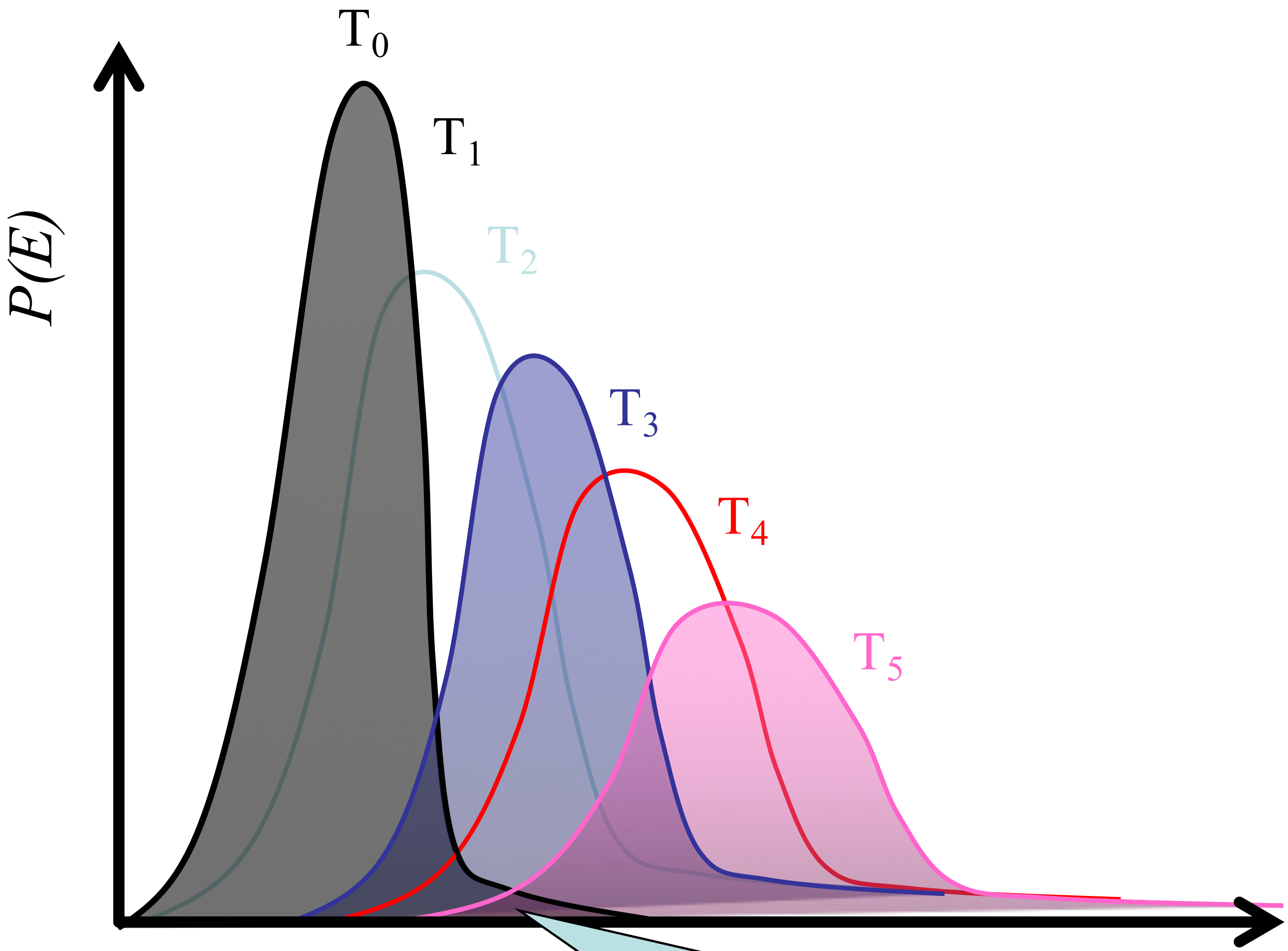
$$= \frac{\int \mathrm{d}\mathrm{r}^N A\left(\mathrm{r}^N\right)\exp\left[-\beta_1 U\left(\mathrm{r}^N\right)\right]\exp\left[\beta_2 U\left(\mathrm{r}^N\right)-\beta_2 U\left(\mathrm{r}^N\right)\right]}{\int \mathrm{d}\mathrm{r}^N \exp\left[-\beta_1 U\left(\mathrm{r}^N\right)\right]\exp\left[\beta_2 U\left(\mathrm{r}^N\right)-\beta_2 U\left(\mathrm{r}^N\right)\right]}$$

$$= \frac{\int \mathrm{d}\mathrm{r}^N A\left(\mathrm{r}^N\right)\exp\left[\beta_2 U\left(\mathrm{r}^N\right)-\beta_1 U\left(\mathrm{r}^N\right)\right]\exp\left[-\beta_2 U\left(\mathrm{r}^N\right)\right]}{\int \mathrm{d}\mathrm{r}^N \exp\left[\beta_2 U\left(\mathrm{r}^N\right)-\beta_1 U\left(\mathrm{r}^N\right)\right]\exp\left[-\beta_2 U\left(\mathrm{r}^N\right)\right]}$$

We only need a *single* simulation!

$$\boxed{= \frac{\left\langle A\exp\left[\left(\beta_2-\beta_1\right)U\right]\right\rangle_{NVT_2}}{\left\langle \exp\left[\left(\beta_2-\beta_1\right)U\right]\right\rangle_{NVT_2}}}$$

We perform a simulation at $T=T_2$ and we determine A at $T=T_1$



Overlap becomes very small

Metadynamics

- Select a set of "relevant" collective variables (S_α)
- MD: Track trajectory $S_\alpha(t)$ over a time interval $[t_0, t_1]$: $\overline{S}[t_0, t_1]$
- Bias by adding repulsion at region near $\overline{S}[t_0, t_1]$
- Continue MD and iterate

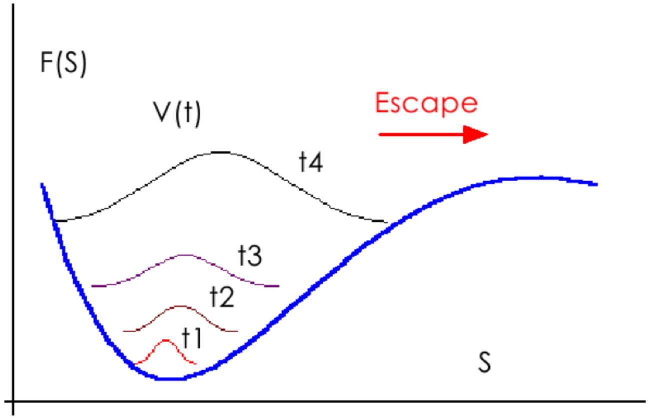


Fig by Jürg Hutter and Marcella Iannuzzi

Example:
Carbon Phase Diagram